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THE CHROMATIC NUMBER OF THE PRODUCT OF TWO 4-CHROMATIC GRAPHS IS 4

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For any graph G and number $n\geqq 1$ two functions f, g from $V(G)$ into $\{1, 2, \ldots, n\}$ are adjacent if for all edges (a, b) of G, $f(a)\neq g(b)$. The graph of all such functions is the colouring graph $\mathcal{C}_n(G)$ of G. We establish first that $\chi(G)=n+1$ implies $\chi(\mathcal{C}_n(G))=n$ iff $\chi(G\times H)=n+1$ for all graphs H with $\chi(H)\geqq n+1$. Then we will prove that indeed for all 4-chromatic graphs G $\chi(\mathcal{C}_3(G))=3$ which establishes Hedetniemi's [3] conjecture for 4-chromatic graphs.

1. Introduction

The product of two graphs $G\times H$ has the vertex set $V(G)\times V(H)$ and edges all pairs $((a, b), (\bar{a}, \bar{b}))$ such that $(a, \bar{a})$ and $(b, \bar{b})$ are edges of G and H, respectively. Observe that if f is a proper colouring of G then the colouring g of $G\times H$ given by $g(a, b)=f(a)$ is a proper colouring of $G\times H$. Hence $\chi(G\times H)\leqq \min(\chi(G), \chi(H))$.

Conjecture 1 (Hedetniemi [3]). *For all G and H and for all $n\geqq 0, \chi(G)>n$ and $\chi(H)>n$ implies $\chi(G\times H)>n$.*

[2] contains some general information on this problem.

Let G be a graph without loops. We define for each positive integer n the n-colouring graph of G, denoted by $\mathcal{C}_n(G)$, as follows. The vertex set of $\mathcal{C}_n(G)$ is the set of all functions $f\colon V(G)\to\{1, \ldots, n\}$ and two such functions f, g are connected by an edge whenever for all edges $ab\in E(G)$, $f(a)\neq g(b)$. This definition allows $\mathcal{C}_n(G)$ to have loops at those vertices which are proper colourings of G. Therefore $\mathcal{C}_n(G)$ has no loops iff $\chi(G)>n$. There are many unanswered questions concerning those colouring graphs but the one closely related to conjecture 1 is to determine the chromatic number of $\mathcal{C}_n(G)$ when $\chi(G)>n$. It is easy to see that $\mathcal{C}_n(G)$, for any graph G, has chromatic number at least n. The constant maps form a complete subgraph of order n.

Conjecture 2. *$\chi(G)>n$ implies that $\chi(\mathcal{C}_n(G))=n$. We will show that Conjecture 1 and Conjecture 2 are equivalent and that Conjecture 2 holds for $n=3$.*

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2. The graph of colourings of a graph

Let G, H be two $(n+1)$-chromatic graphs whose product $G\times H$ is a counter-example to conjecture 1, say $\chi(G\times H)=n$. If $f\colon G\times H\to\{1,\dots,n\}$ is a proper n-colouring then for each vertex $v\in H$ we get an n-colouring f_v of G defined by $f_v(x)=f(x,v)$ $(x\in G)$. We call f_v the colouring of G induced by v. The map $\alpha\colon H\to\mathscr{C}_n(G)$ which maps each $v\in H$ to f_v is an edge-preserving map. Since $\mathscr{C}_n(G)$ has no loops then $\chi(H)\leqq\chi(\alpha(H))$. Therefore $\chi(\mathscr{C}_n(G))>n$. Conversely if G is an $(n+1)$-chromatic graph such that $\chi(\mathscr{C}_n(G))>n$ then $G\times\mathscr{C}_n(G)$ is a counter-example to conjecture 1. A proper colouring of $G\times\mathscr{C}_n(G)$ in n colours is obtained by colouring each ordered pair (x, f) by $f(x)$. This establishes the equivalence of conjectures 1 and 2. We continue this section with some general results.

Theorem 2.1. *Let G be a connected $(n+1)$-chromatic graph. Then $\mathscr{C}_n(G)$ contains a unique complete subgraph of order n, namely the subgraph induced by the constant maps.*

Proof. Let $f_1,\dots,f_n$ denote the vertices of a complete subgraph of $\mathscr{C}_n(G)$. Let H be a vertex-critical $(n+1)$-chromatic subgraph of G. We claim that for each $i=1,\dots,n$ for each $x\in H$ there is a vertex y adjacent to x in H such that $f_i(x)=$ $=f_i(y)$. Assume not, then there are $x\in H$ and a colour, say f_1, such that $f_1(x)\neq$ $\neq f_1(y)$ for each vertex y adjacent to x in H. Since $H-x$ is n-chromatic there is a partition of the vertices of $H-x$ into n independent subsets $V_1,\dots,V_n$. We get a proper n-colouring f of H as follows:

$$f(x)=f_1(x)$$

$$f(y)=f_i(y) \quad \text{where} \quad y\in V_i.$$

This contradiction proves our claim. From this we deduce that for each vertex $x\in H$, $f_i(x)\neq f_j(x)$ whenever $i\neq j$. This implies that $f_i(x)=f_i(y)$ for each i and for each pair of adjacent vertices $x, y\in H$.

Therefore the restrictions of $f_1,\dots,f_n$ to H are the constant maps. By the connectedness of G each f_i must be constant on G. ∎

Corollary 2.2. (D. Duffus, B. Sands, and R. E. Woodrow [2]). *Let G, H be two connected $(n+1)$-chromatic graphs both containing a complete subgraph of order n. Then $G\times H$ is $(n+1)$-chromatic.*

Proof. Denote by $x_1,\dots,x_n$ and by $y_1,\dots,y_n$ the vertices of the complete subgraphs of G and H respectively. Suppose that $f\colon G\times H\to\{1,\dots,n\}$ is a proper n-colouring. Since $\mathscr{C}_n(G)$ has no loops the induced colourings $f_{y_1},\dots,f_{y_n}$ of G are all distinct and form a complete subgraph of order n. By the previous theorem, these induced colourings are the constant maps. In other words for each fixed i, $f(x, y_i)$ is independent of x. In a similar way $f(x_i, y)$ is independent of y for fixed i. However, this is an obvious contradiction. ∎

Corollary 2.3. (Burr, Erdős and Lovász [1]) *Let G be an $(n+1)$-chromatic graph in which each vertex is contained in a complete subgraph of order n. Then $\chi(G\times H)=$ $=n+1$ for each $(n+1)$-chromatic graph H.*

Proof. Suppose that f is a proper n-colouring of $G \times H$. As we noticed earlier, the map α: $G \to \mathscr{C}_n(H)$ defined by $\alpha(x)=f_x$ is edge-preserving. The image under f of a complete subgraph of G must be a complete subgraph of $\mathscr{C}_n(H)$ of the same order since $\mathscr{C}_n(H)$ has no loops. This implies that α maps G onto the complete subgraph of the constant maps of H. This is a contradiction since G is $(n+1)$-chromatic. ▮

Corollary 2.4. (Hedetniemi [3]) *If* $\chi(G) \geqq 3$ *and G is connected, then* $\mathscr{C}_2(G)$ *contains exactly one edge, hence* $\chi(\mathscr{C}_2(G))=2$. *So* $\chi(G \times H)=3$ *for any two 3-chromatic graphs G and H.* ▮

3. The 3-colouring graph of an odd circuit

In this section, C_n will denote a circuit on n vertices $v_1, \ldots, v_n$ with edges $v_i v_{i+1}$ where $v_{n+1}=v_1$. To obtain our main result we are interested in the case where n is odd but include the even case for completeness.

Let $f \in \mathscr{C}_3(C_n)$. A vertex $v_i \in C_n$ is defined to be a *fixed* vertex for f, or fixed by f, if its two neighbours get different colours, that is when $f(v_{i-1}) \neq f(v_{i+1})$. The reason for the term fixed is that if v_i is fixed by f then $g(v_i)$ has the same value for all maps $g \in \mathscr{C}_3(C_n)$ adjacent to f. We say that f has an odd parity, or simply f is an *odd* colouring, when it has an odd number of fixed points. Similarly, colourings with even parity are defined.

Lemma 3.1. *Let* $f \in \mathscr{C}_3(C_n)$. *Then the number of triples of consecutive vertices* v_{i-1}, v_i, v_{i+1} *which get three different colours by f has the same parity as f itself.*

Proof. Clearly we can assume that f is not a constant map. Partition C_n into monochromatic intervals of consecutive vertices. The contribution to the number of fixed vertices from a monochromatic interval $\{v_i, \ldots, v_{i+k}\}$ $(k \geqq 1)$ is two since the only fixed vertices in this interval are the endvertices v_i, v_{i+k}. An interval of a single vertex $\{v_i\}$ contributes one if and only if v_{i-1}, v_i, v_{i+1} get three different colours. ▮

Lemma 3.2. *A proper colouring of an odd (resp. even) circuit with at most three colours is odd (resp. even).*

Proof. We use induction on the length of the circuit. A proper colouring of a triangle has three fixed vertices. A proper colouring of a quadrilateral has no fixed vertices or two fixed ones depending on whether it uses two or three colours. Let f be a proper colouring of C_n $(n \geqq 5)$. The statement is true if each vertex is fixed by f. So, without loss of generality, we assume that v_n is not fixed by f and let $f(v_{n-1})=f(v_1)=1$ and $f(v_n)=2$. If we remove v_n and identify v_{n-1} and v_1 we get an $(n-2)$-circuit which is still properly coloured by $v_i \to f(v_i)$. It is easy to check that the number of fixed vertices decreases by two if $f(v_2)=f(v_{n-2})=3$ and does not change in all other cases. Therefore, the parity of f is the same as the parity of the resulting proper colouring of C_{n-2}. This completes the proof of the induction step and the lemma. ▮

Lemma 3.3. *Let* f_1 *and* f_2 *be connected by an edge of* $\mathscr{C}_3(C_n)$. *Then* f_1, f_2 *have the same parity.*

Proof. The graph $C_n \times K_2$ consists of a $2n$-circuit when n is odd and two n-circuits when n is even. Denote by a_1, a_2 the vertices of K_2 and define a proper colouring f of $C_n \times K_2$ by $f(v_i, a_j) = f_j(v_i)$, $i = 1, \dots, n$; $j = 1, 2$. A vertex (v_i, a_j) is fixed by f if and only if v_i is fixed by $f_{j'}$ where $j' \in \{1, 2\} \setminus \{j\}$. This shows that the sum of the number of vertices fixed by f_1 and f_2 is equal to the number of vertices fixed by f. By lemma 3.2, this is even since f is proper. ▮

A consequence of Lemma 3.3 is that the parity is the same for all vertices of a component of $\mathscr{C}_3(C_n)$. Therefore we can speak of the parity of a component of $\mathscr{C}_3(C_n)$.

Proposition 3.4. *Let C_n with vertices $v_1, v_2, \dots, v_n$ and C_m with vertices $u_1, u_2, \dots, u_m$ be two odd circuits. Then for any proper 3-colouring f of $C_n \times C_m$ the parity of the induced colourings f_{v_i} is different from the parity of f_{u_j}.*

Proof. Denote by M_i, N_j the number of vertices fixed respectively by the induced colourings f_{v_i}, f_{u_j} $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$. By Lemma 3.3, all the M_i's are of the same parity and so are the N_j's. To show that the M_i's and the N_j's have different parities, it suffices to prove that the number

$$nm - \sum_{i=1}^{n} M_i - \sum_{j=1}^{m} N_j$$

is even. To this end we investigate how the quadrilaterals in $C_n \times C_m$ are coloured by f. Let Q_{ij} denote the quadrilateral with vertices (v_{i-1}, u_j), (v_i, u_{j+1}), (v_{i+1}, u_j) and (v_i, u_{j-1}). Since f uses only three colours, there are exactly three possible cases:

(i) $f(v_{i-1}, u_j) \neq f(v_{i+1}, u_j)$,

(ii) $f(v_i, u_{j-1}) \neq f(v_i, u_{j+1})$,

(iii) $f(v_{i-1}, u_j) = f(v_{i+1}, u_j)$ and $f(v_i, u_{j-1}) = f(v_i, u_{j+1})$.

These cases correspond respectively to: vertex v_i is fixed by colour f_{u_j}, vertex u_j is fixed by colour f_{v_i} and neither is fixed. Here it is not possible that v_i, u_j are both fixed respectively by f_{u_j}, f_{v_i}. Therefore the number $nm - \sum_{i=1}^{n} M_i - \sum_{j=1}^{m} N_j$ is equal to the number of the quadrilaterals Q_{ij} such that (iii) holds. Observe that (iii) holds if and only if the vertices of Q_{ij} are coloured by two colours only. We finish the proof by showing that the number of 2-coloured quadrilaterals is even. Direct the edges of $C_n \times C_m$ such that arrows go from colours 1 to 2, 2 to 3 and from 3 to 1. It is not difficult to check that opposite edges in Q_{ij} have the same "sense" whenever it is coloured by three colours or, equivalently, when either (i) or (ii) holds. But if Q_{ij} gets only two colours, that is (iii) holds, then its opposite edges have different "senses". See Figure 1.

Consider the sequence of quadrilaterals $Q_{11}, Q_{22}, \dots, Q_{ij}, Q_{i+1, j+1}, \dots, Q_{11}$. Any two consecutive quadrilaterals in this sequence have an edge in common. The number of 2-coloured quadrilaterals in this sequence is even since it equals the number of reversals in the "sense" of the common edge. The set of all quadrilaterals can be partitionned into k disjoint such sequences where $k = \text{g.c.d.}\ (m, n)$. This shows that the total number of 2-coloured quadrilaterials is even as required. ▮

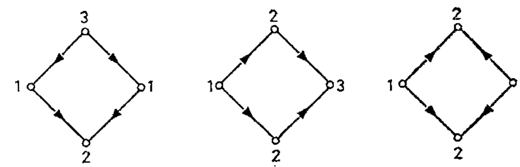

Fig. 1

4. The 3-colouring graph of a 4-chromatic graph

In this section we prove our main result that $\mathscr{C}_3(G)$ is 3-chromatic for all 4-chromatic graphs G. In order to prove this, we consider the restriction of colourings $f\in\mathscr{C}_3(G)$ to odd circuits of G. We were not able to find an example of a 4-chromatic graph G and a colouring $f\in\mathscr{C}_3(G)$ whose restriction to each odd circuit of G has an odd parity. It is likely that such a colouring cannot exist. However, the following proposition shows that if it exists, then it is an isolated vertex of $\mathscr{C}_3(G)$.

Proposition 4.1. *Let G be a 4-chromatic graph. Suppose there is a colouring $f\in\mathscr{C}_3(G)$ whose restriction to each odd circuit in G has an odd parity. Then f is an isolated vertex of $\mathscr{C}_3(G)$.*

Proof. Assume fg is an edge of $\mathscr{C}_3(G)$. Define

$$X = \{x\in V(G): \exists y\in V(G) \text{ with } xy\in E(G) \text{ and } f(x)=f(y)\}.$$

We claim that the induced subgraph $G(X)$ has chromatic number at least three. Obviously $\chi(G(X))\geqq 2$. Suppose $G(X)$ is 2-chromatic and let $X=X_1\cup X_2$ be a partition into two colour classes. We get a proper 3-colouring h of G defined by

$$h(v)=\begin{cases} f(v) & \text{for } v\in V(G)-X_1\\ g(v) & \text{for } v\in X_1\end{cases}$$

which is a contradiction. Therefore $\chi(G(X))\geqq 3$ and $G(X)$ contains an odd circuit which we denote by C. By Lemma 3.1 there is a consecutive triple of vertices v_1, v_2, v_3 on C with $\{f(v_1), f(v_2), f(v_3)\}=\{1, 2, 3\}$, say $f(v_i)=i$. This implies that $g(v_2)=2$. By the definition of X, there is a vertex $u\in G$ adjacent to v_2 such that $f(u)=f(v_2)=2$. Therefore $f(u)=g(v_2)$ which is a contradiction. ▮

Theorem 4.2. *Let C_n be an odd circuit. Then each component of $\mathscr{C}_3(C_n)$ with even parity is at most 3-chromatic.*

Proof. Let T be an even-parity component of $\mathscr{C}_3(C_n)$ and assume that H is a connected 4-chromatic subgraph of T. Define a 3-colouring φ of the graph $C_n\times H$ by $\varphi(v, h)=h(v)$. φ is a proper colouring of $C_n\times H$ and for each $h\in H$ the induced colouring φ_h is the colouring h itself. By Proposition 3.4 each induced colouring φ_v $(v\in C_n)$ must have odd parity on each odd circuit of H. Moreover, two such induced colouring $\varphi_v, \varphi_{v'}$ are adjacent in $\mathscr{C}_3(H)$ whenever v, v' are adjacent in C. This contradicts Proposition 4.1. ▮

Theorem 4.3. *$\mathscr{C}_3(G)$ is 3-chromatic for each 4-chromatic graph G.*

Proof. Let H be a connected 4-chromatic subgraph of $\mathscr{C}_3(G)$ and $h_1 \in H$. From Proposition 4.1, there must exist an odd circuit C in G such that the restriction of h_1 to C has an even parity. Define a map $\alpha: H \to \mathscr{C}_3(C)$ by mapping each colouring $h \in H$ to its restriction to C. It is clear that α is edge-preserving. Therefore α maps H into a component T of $\mathscr{C}_3(C)$ with even parity. This component has no loops since it contains no proper colouring of C. Therefore $\chi(H) \leqq \chi(T)$, in contradiction to Theorem 4.2. ∎

Suppose G_1, G_2 are connected 4-chromatic graphs. Let C_i be any two odd circuits in G_i $(i=1, 2)$. Denote by H the subgraph $(G_1 \times C_2) \cup (C_1 \times G_2)$ of $G_1 \times G_2$. H is 4-chromatic. To prove this suppose that f is a proper 3-colouring of H. As in the proof of Theorem 4.3 we get two edge preserving maps $\alpha: G_1 \to \mathscr{C}_3(C_2)$ and $\beta: G_2 \to \mathscr{C}_3(C_1)$. The image of one of these maps must lie in an even-parity component of the corresponding colouring graph which is a contradiction. This shows that even if G_1, G_2 were critical, $G_1 \times G_2$ is far from being critical. In fact any one of its vertices can be removed and still we have a 4-chromatic graph. If one expects this to be true in general, then one has the following conjecture.

Conjecture 3. *Let G_1, G_2 be connected n-chromatic graphs. For $i=1, 2$ let H_i be an $(n-1)$-chromatic subgraph of G_i. Then the subgraph $(G_1 \times H_2) \cup (H_1 \times G_2)$ of $G_1 \times G_2$ is n-chromatic.*

A feature of the proof of Corollary 2.2 is that this conjecture is true when each H_i is the complete graph K_{n-1}.

References

[1] S. A. Burr, P. Erdős and L. Lovász, On graphs of Ramsey type, *Ars Comb.* **1** (1976), 167—190.
[2] D. Duffus, B. Sands and R. E. Woodrow, On the Chromatic Number of the Product of Graphs, *Journal of Graph Theory, to appear.*
[3] S. T. Hedetniemi, Homomorphisms of graphs and automata, *Univ. of Michigan Technical Report 03105-44-T, 1966.*

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